Econ 802
Answers to Second Midterm

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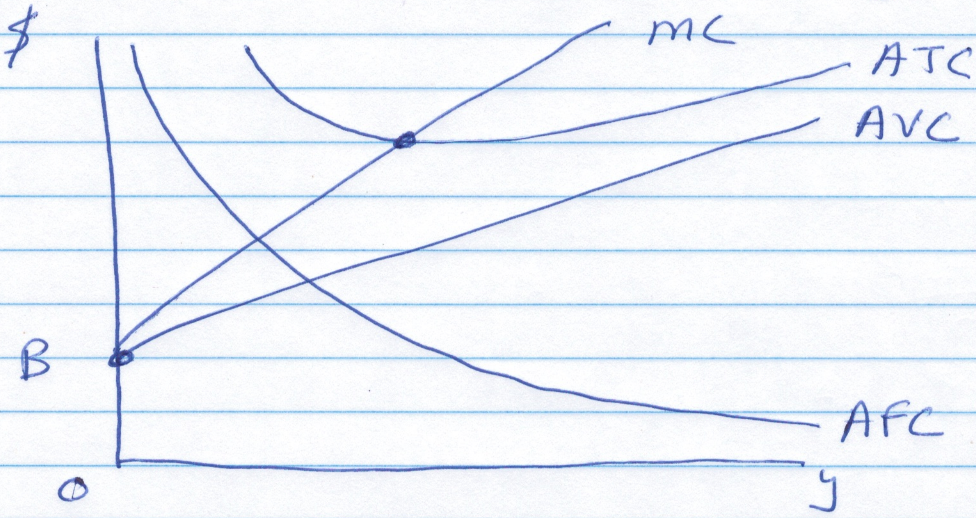
1(a) First divide costs into fixed (A) and variable ($By + Cy^2$). Average fixed cost is $\frac{A}{y}$

Average variable cost is $B + Cy$

Average total cost is $\frac{A}{y} + B + Cy$

Marginal cost is $B + 2Cy$



AFC is always falling and $\to 0$ as $y \to \infty$

AVC and MC are linear, have the same vertical intercept, and MC is twice as steep as AVC.

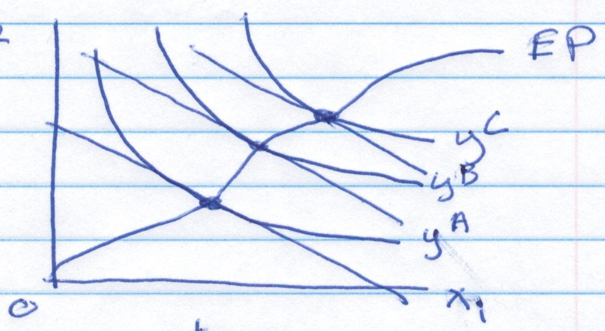
ATC is U-shaped (you can do some calculus to confirm this) and MC passes through its minimum point

Also, ATC is above AVC but the gap between them falls as AFC $\to 0$ at large outputs.

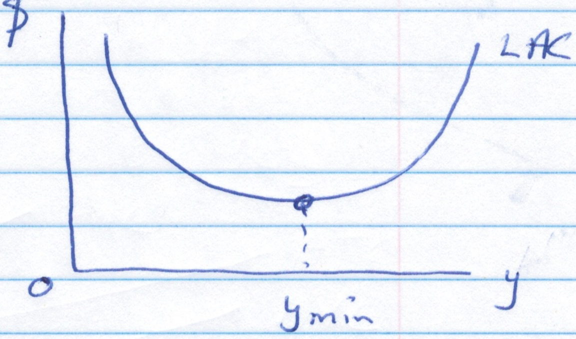
(b) For fixed input prices w, the expansion path is the set of all points that are the cost-minimizing way to produce some output level $y > 0$.

Graphically:

(The slopes of the isocost lines are determined by w)



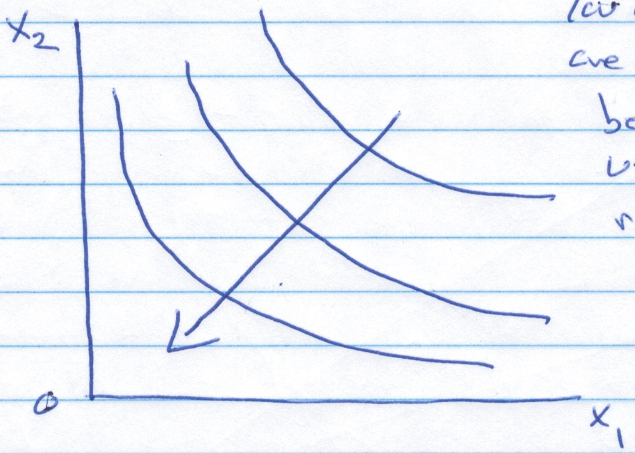
The LAC is U-shaped:



For any $y < y_{min}$ the firm will operate on the corresponding isoquant at a point on the EP. We have shown that when LAC is falling, the local elasticity with respect to scale is greater than one (local IRS) at the cost minimizing input vector, so $e(x) > 1$ for this point on EP. For the isoquant corresponding to y_{min} the firm operates at the point on EP that minimizes cost for y_{min} (call this point x_{min}). Then we have $e(x_{min}) = 1$ (local CRS). When LAC is rising, the firm will operate at points on EP where $e(x) < 1$ (local DRS).

1(c) Professor X is arguing that all firms have global CRS. S/he is ignoring several possibilities. First, there may be indivisibilities in some inputs that make it impossible to scale all activities up or down in a continuous way. For example, it may be impossible to have half a factory. Also, having more of all inputs may make it possible to use new techniques that were not feasible before, yielding IRS. Or even in the long run, some inputs supplied by nature (like sun or rain) cannot be increased, yielding DRS. (You could also use organizational arguments about division of labor, coordination problems at large scales etc.)

2(a) The important thing to notice is that the utility function is just a Cobb-Douglas function with a minus sign in front. If we hold u constant we can write $-u = x_1^a x_2^b$ so the indifference curves must have the same shape as the isoquants for a C-D production function:



You can verify that the ICs are downward sloping and bend toward the origin using calculus: the marginal rate of substitution is

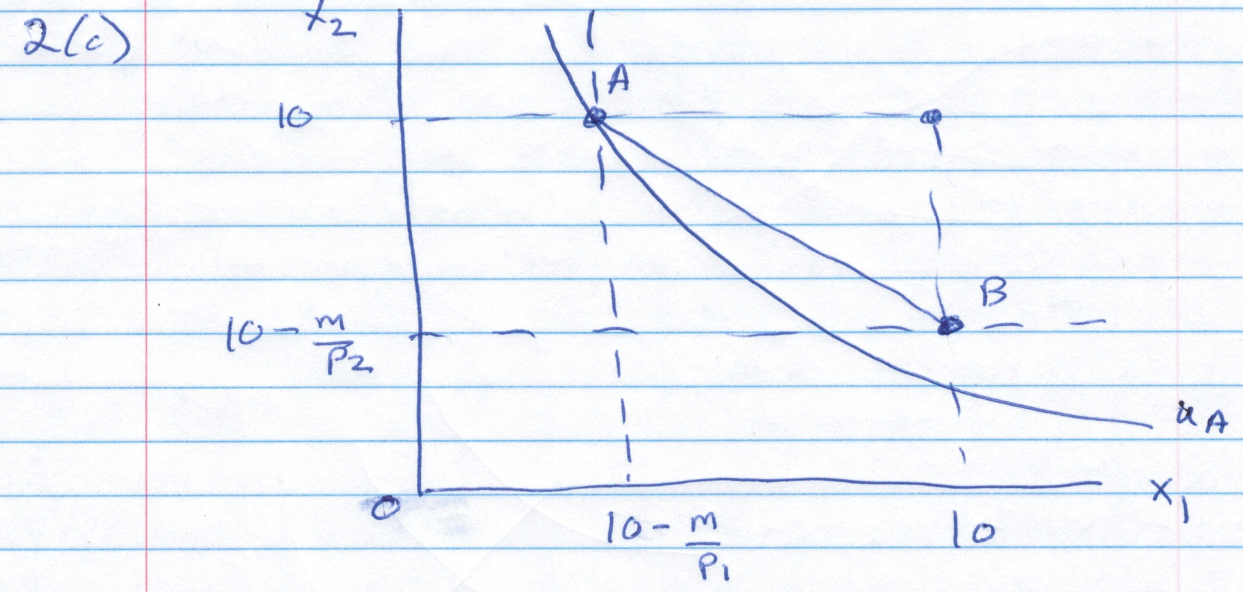
$$MRS = -\frac{MU_1}{MU_2}$$

where MU_i is the marginal utility of good i.

The only real difference from a Cobb-Douglas case is that here x_1 and x_2 are bads rather than goods so utility is increasing as we get closer to the origin.

(b) The highest possible utility is zero. This occurs when either $x_1 = 0$ or $x_2 = 0$ or both. Otherwise utility is negative. One way to get zero utility is $t_1 = x_1 = 10$ which costs $10p_1$. If $10p_1 < m$ or $10 < \frac{m}{p_1}$ then there is no need to spend all of m in order to get zero utility; less than m will be sufficient. The same is true if $10p_2 < m$ or $10 < \frac{m}{p_2}$. If $10p_1 = m$ or $10p_2 = m$, it is possible to achieve zero utility but it is necessary to spend all of m to do it. If $10p_1 > m$ and $10p_2 > m$, every point on the constraint line $p_1 t_1 + p_2 t_2 = m$ leaves $x_1 > 0$ and $x_2 > 0$. In this case Mr. Clean still has some dirt and trash even when he spends all of his income. He would like more cleaning services but can't afford it.

2(c)



If Mr. Clean puts all of his income into t_1, he will have $x_1 = 10 - \frac{m}{p_1}$ and $x_2 = 10$. If he puts all of his income into t_2, he will have $x_1 = 10$ and $x_2 = 10 - \frac{m}{p_2}$. The budget constraint is $p_1 t_1 + p_2 t_2 = m$ with $x_1 = 10 - t_1$ and $x_2 = 10 - t_2$ so $t_1 = 10 - x_1$ and $t_2 = 10 - x_2$. Substitution gives the constraint $p_1(10 - x_1) + p_2(10 - x_2) = m$ or $p_1 x_1 + p_2 x_2 = 10(p_1 + p_2) - m$. This corresponds to the downward sloping line segment AB. Given the curvature of the indiff. curves and the fact that utility increases as we approach the origin, we will not have a tangency solution. Instead we have a boundary solution like the one shown at point A. Depending on the prices, the solution could occur at point B instead. But in general, Mr Clean spends all of his income either on removing dirt or removing trash, not some combination of the two.

3(a) We have $v(p,m) = \dfrac{m}{\sum_i \left(\frac{p_i}{a_i}\right)}$

To compute the Marshallian demands, use Roy's Identity:

$$x_i(p,m) = -\frac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}} = -\frac{\frac{m}{\left(\sum \frac{p_i}{a_i}\right)^2}(-1)\left(\frac{1}{a_i}\right)}{\frac{1}{\left(\sum \frac{p_i}{a_i}\right)}} = \frac{m}{a_i}\frac{1}{\sum_i \frac{p_i}{a_i}}$$

(all $i = 1 \ldots n$)

For the Hicksian demands first invert the indirect utility function to get the expenditure function:

$$u = \frac{m}{\sum \frac{p_i}{a_i}} \Rightarrow m = u \sum \frac{p_i}{a_i} \Rightarrow e(p,u) = u \sum \frac{p_i}{a_i}$$

Then use Shephard's Lemma:

$$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i} = \frac{u}{a_i} \quad (\text{all } i = 1 \ldots n)$$

(b) The Slutsky equation says

$$\frac{\partial x_j(p,m)}{\partial p_i} = \frac{\partial h_j[p, v(p,m)]}{\partial p_i} - \frac{\partial x_j(p,m)}{\partial m} \cdot x_i(p,m)$$

Let's confirm that the demands in part (a) satisfy this.

$$\frac{\partial x_j}{\partial p_i} = \frac{m}{a_j}(-1)\frac{1}{\left(\sum \frac{p_i}{a_i}\right)^2}\left(\frac{1}{a_i}\right) = -\frac{m}{a_i a_j}\frac{1}{\left(\sum \frac{p_i}{a_i}\right)^2}$$

$$\frac{\partial h_j}{\partial p_i} = 0 \,; \quad \frac{\partial x_j}{\partial m} = \frac{1}{a_j}\left(\frac{1}{\sum \frac{p_i}{a_i}}\right); \quad x_i = \frac{m}{a_i}\left(\frac{1}{\sum \frac{p_i}{a_i}}\right)$$

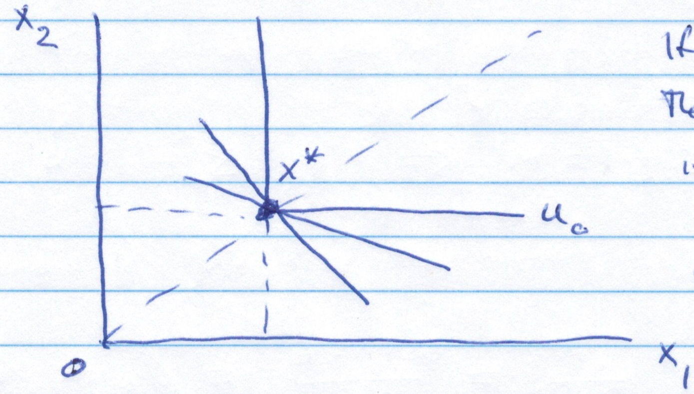
So we have

$$-\frac{m}{a_i a_j}\frac{1}{\left(\sum \frac{p_i}{a_i}\right)^2} = 0 - \frac{1}{a_j}\left(\frac{1}{\sum \frac{p_i}{a_i}}\right)\frac{m}{a_i}\left(\frac{1}{\sum \frac{p_i}{a_i}}\right)$$

which is true.

(6)

3(c) A good guess would be a Leontief utility function of the form $u(x) = \min\{a_1 x_1 \ldots a_i x_i \ldots a_n x_n\}$ The reason is that the Hicksian demands in part (a) do not depend on the prices, so there are no substitution effects. This is consistent with a graphical analysis of the two good case: $u = \min\{a_1 x_1, a_2 x_2\}$



If we want to minimize the expenditure for the indiff curve u_0 we will always choose the corner point x^* no matter what the prices are.

To confirm that this guess is correct, let's minimize expenditure in the n-good case. Fix the utility level u. We need $a_1 x_1 = \cdots a_i x_i = \cdots a_n x_n = u$ for all i. If any x_i is smaller, we don't achieve u, and if any x_i is larger, we are spending money unnecessarily (it would be cheaper to reduce x_i slightly). So the Hicksian demands must be obtained from $a_i x_i = u$ for all $i = 1..n$ or $x_i = h_i(p,u) = \frac{u}{a_i}$

This gives the expenditure function $e(p,u) = \sum_i \frac{p_i u}{a_i} = u \sum_i \frac{p_i}{a_i}$ which is what we had in part (a). Inverting this gives the indirect utility function.

Note: if you try to use the method $u(x) = \min_p v(p,1)$ subject to $px = 1$ the prices will drop out of the FOC. The problem is that the direct utility function is not differentiable and the inverse demand functions are not well defined.

4(a) From the budget constraint $pc = wH + r$ we have

$pc = w(T-L) + r \Rightarrow pc + wL = \underbrace{wT + r}_{\text{call this } m.}$

Set up the usual Lagrangean to find the Marshallian demands:

$$\mathcal{L} = \ln c + \ln L - \lambda[pc + wL - m]$$

FOC: $\left.\begin{aligned} \frac{1}{c} - \lambda p = 0 \\ \frac{1}{L} - \lambda w = 0 \end{aligned}\right\} \Rightarrow \frac{\frac{1}{c}}{\frac{1}{L}} = \frac{p}{w}$ or $\frac{L}{c} = \frac{p}{w}$

Substitute into the constraint: $pc + w\left(\frac{pc}{w}\right) = m$

$\Rightarrow 2pc = m \Rightarrow \begin{cases} c = \frac{m}{2p} \\ L = \frac{m}{2w} \end{cases}$ so $c(p,w,r) = \frac{wT+r}{2p}$

$L(p,w,r) = \frac{wT+r}{2w}$

Note: The utility function is a log transformation of a Cobb-Douglas function which we know is strictly quasi-concave. Therefore we don't need to check SOC.

(b) We could use either Hicksian or functional separability. The Hicksian approach is easier so I will use it here. Write the consumption bundle in the form $(L, c_1 \ldots c_k)$ where L is leisure. Let the prices be $(w, p_1 \ldots p_k)$. Assume the vector $p = (p_1 \ldots p_k)$ always satisfies $p = t p_0$ for some fixed vector p_0 with $t > 0$, so the relative prices of the consumption goods don't change. We can write indirect utility as

$$v(w, t, m) = \max_{(L,c)} u(L, c_1 \ldots c_k) \text{ s.t. } wL + t p_0 c = m$$

where $c = (c_1 \ldots c_k)$ is a vector.

Think of $(p_0 c)$ as a composite consumption good whose price is t. Define big $C = p_0 c$ and go from the indirect utility function $v(w,t,m)$ back to a direct utility function involving L and C:

$$w(L, C) = \min_{(w,t)} v(w, t, 1) \text{ subject to } wL + tC = 1$$

This gives a direct utility function That depends only on leisure (L) and The composite consumption good (C).

4 (c) From The answers in part (a) we have

$$c_i = \frac{m_i}{2p} \quad \text{and} \quad L_i = \frac{m_i}{2w}$$

$$\text{so } c_i = \frac{wT + r_i}{2p} \quad \text{and} \quad L_i = \frac{wT + r_i}{2w}$$

Summing over consumers, $C(p, w, R) = \dfrac{nwT + R}{2p}$

$$L(p, w, R) = \frac{nwT + R}{2w}$$

Now substitute The individual demands into $u_i = \ln c_i + \ln L_i$ to get The indirect utility

$$v(p, m_i) = \ln\left(\frac{m_i}{2p}\right) + \ln\left(\frac{m_i}{2w}\right) = 2 \ln m_i - \ln 2p - \ln 2w$$

Due to The non-linearity in $m_i = wT + r_i$, we cannot add up $v(p, m_i)$ across consumers to get an aggregate indirect utility That depends only on $M = \sum m_i$ or $R = \sum_i r_i$. Furthermore $v(p, m_i)$ is not in The Gorman form $a_i(p) + b(p) m_i$ because it is not linear in m_i.

However, we can use any monotonic transformation of The original direct utility function. Consider $u(c_i, L_i) = (c_i L_i)^{1/2}$ which represents The same preferences (if you square This and take logs, you get $\ln c_i + \ln L_i$)

Plug The Marshallian demands into This transformed utility function to get $v(p, m_i) = \left[\left(\frac{m_i}{2p}\right)\left(\frac{m_i}{2w}\right)\right]^{1/2} = \dfrac{m_i}{2p^{1/2} w^{1/2}}$

This is in The Gorman form, where $a_i(p) = 0$ and $b(p) = \dfrac{1}{2p^{1/2} w^{1/2}}$.

Thus aggregation becomes possible.

5(a) It is true that we cannot say for certain whether Marshallian demand curves slope up or down. But the theory makes some definite predictions that could be tested using data. The matrix version of the Slutsky equation is

$$\frac{\partial x(p,m)}{\partial p} = \frac{\partial h[p, v(p,m)]}{\partial p} - \frac{\partial x(p,m)}{\partial m} \cdot x(p,m)$$

or

$$\frac{\partial x(p,m)}{\partial p} + \frac{\partial x(p,m)}{\partial m} \cdot x(p,m) = \frac{\partial h[p, v(p,m)]}{\partial p}$$

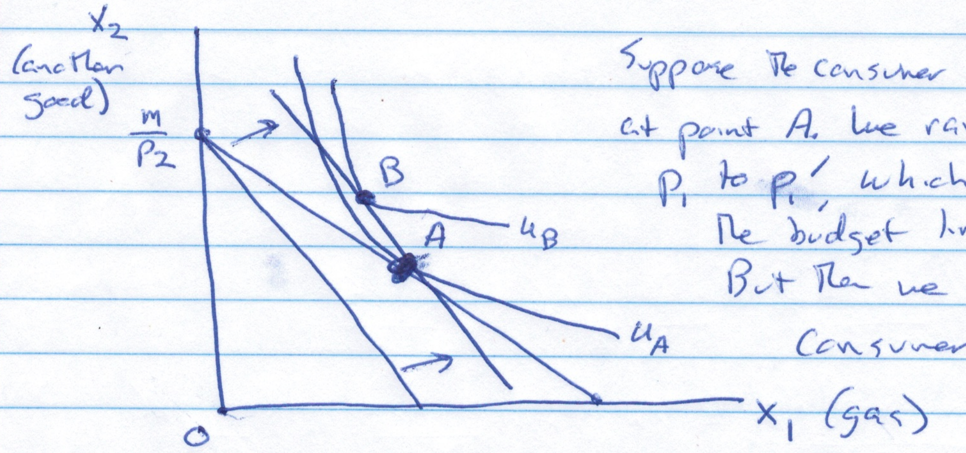
Also, Shephard's Lemma gives $h(p,u) = \frac{\partial e(p,u)}{\partial p}$ where the expenditure function is concave in prices. Therefore the Hessian $\frac{\partial^2 e(p,u)}{\partial p^2} = \frac{\partial h(p,u)}{\partial p}$ is negative semi-definite (and symmetric). This implies that the matrix

$$\frac{\partial x(p,m)}{\partial p} + \frac{\partial x(p,m)}{\partial m} \cdot x(p,m)$$ is symmetric and negative semi-def.

If we had data on p, m, and x, we could estimate the demand functions and test whether these predictions are accurate.

(b) Even if we refund the tax revenue to the consumer and this makes it possible for the consumer to purchase their previous consumption bundle the change in relative prices will generally influence behavior, and it will generally be true that the consumer buys less gas than before.

To see this consider the following graph.



Suppose the consumer is initially at point A. We raise the price p_1 to p_1' which makes the budget line steeper. But then we give the consumer enough additional income that they can afford the previous bundle A at the new prices (indicated by the parallel shift in the budget line). Given the new price ratio, the consumer prefers point B to point A, and therefore consumes less gas, even though it is feasible to consume as much as before.

5(c) The first two sentences are correct. Firms do minimize cost (this is necessary for profit max), and consumers do minimize expenditure (duality says that the same consumption bundle minimizes expenditure and maximizes utility simultaneously).

However, the theories are not identical because we do not impose a budget constraint on the firm (we do not assume that a firm maximizes output subject to an expenditure constraint). We do impose a budget constraint on the consumer.

This leads to important differences in the two theories. For example, we often have to worry about whether there is a solution to a firm's profit max problem, while we rarely worry about this for a utility max problem. We also have the Slutsky equation for the consumer, but there is no similar equation for the firm.